

LOCAL CONTRIBUTIONS TO DONALDSON-THOMAS INVARIANTS

ANDREA T. RICOLFI

ABSTRACT. Let C be a smooth curve embedded in a smooth quasi-projective threefold Y , and let $Q_C^n = \text{Quot}_n(\mathcal{I}_C)$ be the Quot scheme of length n quotients of its ideal sheaf. We show the identity $\tilde{\chi}(Q_C^n) = (-1)^n \chi(Q_C^n)$, where $\tilde{\chi}$ is the Behrend weighted Euler characteristic. When Y is a projective Calabi-Yau threefold, this shows that the DT contribution of a smooth rigid curve is the signed Euler characteristic of the moduli space. This can be rephrased as a DT/PT wall-crossing type formula, which can be formulated for arbitrary smooth curves. In general, the formula is shown to be equivalent to a certain Behrend function identity.

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1. INTRODUCTION

One of the conjectures in [MNOP06] stated that 0-dimensional Donaldson-Thomas (DT) invariants of a smooth projective Calabi-Yau threefold equal the signed Euler characteristic of the moduli space. Now, the more general formula

$$(1.1) \quad \tilde{\chi}(\text{Hilb}^n Y) = (-1)^n \chi(\text{Hilb}^n Y)$$

is known to hold for *any* smooth threefold Y , proper or not [BF08, Thm. 4.11]. Here $\tilde{\chi} = \chi(-, \nu)$ is the Euler characteristic weighted by the Behrend function [Beh09]. The 0-dimensional MNOP conjecture is also solved with cobordism techniques in [Li06, LP09].

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1.1. Main result. We propose a statement analogous to (1.1), again with no Calabi-Yau or properness assumption on the threefold Y , but where a curve is present. More precisely, we focus on the space of 1-dimensional subschemes $Z \subset Y$ whose fundamental class is the cycle of a fixed Cohen-Macaulay curve $C \subset Y$. A natural scheme structure on this space seems to be provided by the Quot scheme

$$Q_C^n = \text{Quot}_n(\mathcal{I}_C)$$

of 0-dimensional length n quotients of \mathcal{I}_C , the ideal sheaf of C . By identifying a surjection $\mathcal{I}_C \twoheadrightarrow F$ with its kernel \mathcal{I}_Z , we see that Q_C^n parametrizes curves $Z \subset Y$ differing from C by a finite subscheme of length n . Our main result, proved in § 4, is the following weighted Euler characteristic computation.

THEOREM. *Let Y be a smooth quasi-projective threefold, $C \subset Y$ a smooth curve. If $Q_C^n = \text{Quot}_n(\mathcal{I}_C)$, then*

$$(1.2) \quad \tilde{\chi}(Q_C^n) = (-1)^n \chi(Q_C^n).$$

The proof uses stratification techniques as in [BF08] and [BB07].

1.2. Applications. Let Y be a smooth projective threefold. Let $I_m(Y, \beta)$ be the Hilbert scheme of curves $Z \subset Y$ in class $\beta \in H_2(Y, \mathbb{Z})$, with $\chi(\mathcal{O}_Z) = m$. Given a Cohen-Macaulay curve $C \subset Y$ of arithmetic genus g , embedded in class β , we show there is a closed immersion $\iota : Q_C^n \rightarrow I_{1-g+n}(Y, \beta)$. We define

$$(1.3) \quad I_n(Y, C) \subset I_{1-g+n}(Y, \beta) = I$$

to be its scheme-theoretic image. When Y is Calabi-Yau, we define the *contribution* of C to the full (degree β) DT invariant of I to be the weighted Euler characteristic

$$(1.4) \quad \text{DT}_{n,C} = \chi(I_n(Y, C), \nu_I).$$

A first consequence of (1.2) is the identity

$$\text{DT}_{n,C} = (-1)^n \chi(I_n(Y, C))$$

when C is a smooth *rigid* curve in Y , because in this case (1.3) is both open and closed.

1.2.1. Local DT/PT correspondence. Let $P_m(Y, \beta)$ be the moduli space of stable pairs introduced by Pandharipande and Thomas [PT09]. For a Calabi-Yau threefold Y and a homology class $\beta \in H_2(Y, \mathbb{Z})$, the generating functions encoding the DT and PT invariants of Y satisfy the “wall-crossing type” formula

$$\text{DT}_\beta(Y, q) = M(-q)^{\chi(Y)} \cdot \text{PT}_\beta(Y, q).$$

Here and throughout, $M(q)$ denotes the MacMahon function, the generating series of plane partitions, that is,

$$M(q) = \sum_{\pi} q^{|\pi|} = \prod_{k \geq 1} (1 - q^k)^{-k}.$$

The DT/PT correspondence stated above was first conjectured in [PT09] and later proved in [Bri11]. In this paper we ask about a similar formula relating the *local* invariants, that is, the contributions of a single smooth curve $C \subset Y$ to the full DT and PT invariants of Y in the class $\beta = [C]$.

If $C \subset Y$ is a fixed smooth curve of genus g , we consider the closed subscheme

$$P_n(Y, C) \subset P_{1-g+n}(Y, \beta) = P$$

of stable pairs with Cohen-Macaulay support equal to C . We use (1.2) and the isomorphism $P_n(Y, C) \cong \text{Sym}^n C$ to show the generating function identity

$$(1.5) \quad \sum_{n \geq 0} \tilde{\chi}(I_n(Y, C)) q^n = M(-q)^{\chi(Y)} (1+q)^{2g-2},$$

which holds without any Calabi-Yau assumption.

For Y a Calabi-Yau threefold, we consider the stable pair local contributions

$$\text{PT}_{n,C} = \chi(P_n(Y, C), \nu_P)$$

like we did in (1.4) for ideal sheaves. We assemble all the local invariants into generating functions

$$\begin{aligned} \text{DT}_C(q) &= \sum_{n \geq 0} \text{DT}_{n,C} q^n \\ \text{PT}_C(q) &= \sum_{n \geq 0} \text{PT}_{n,C} q^n. \end{aligned}$$

The PT side has been computed [PT10, Lemma 3.4] and the result is

$$\text{PT}_C(q) = n_{g,C} \cdot (1+q)^{2g-2},$$

where $n_{g,C}$ is the BPS number of C . Therefore it is clear by looking at (1.5) that the DT/PT correspondence

$$(1.6) \quad \text{DT}_C(q) = M(-q)^{\chi(Y)} \cdot \text{PT}_C(q)$$

holds for C if and only if, for every n , one has

$$\text{DT}_{n,C} = n_{g,C} \cdot \tilde{\chi}(I_n(Y, C)).$$

For instance, it holds when C is rigid. In the last section, we discuss the plausibility to conjecture the identity (1.6) to hold for all smooth curves.

Conventions. All schemes are defined over \mathbb{C} , and all threefolds are assumed to be smooth. An *ideal sheaf* is a torsion-free sheaf with rank one and trivial determinant. For a smooth projective threefold Y , we denote by $I_m(Y, \beta)$ the moduli space of ideal sheaves with Chern character $(1, 0, -\beta, -m + \beta \cdot c_1(Y)/2)$. It is naturally isomorphic to the Hilbert scheme parametrizing closed subschemes $Z \subset Y$ of codimension at least 2, with homology class β and $\chi(\mathcal{O}_Z) = m$. The *Calabi-Yau* condition for us is simply the existence of a trivialization $\omega_Y \cong \mathcal{O}_Y$. We use the word *rigid* as a shorthand for the more correct *infinitesimally rigid*: for a smooth embedded curve $C \subset Y$, this means $H^0(C, N_{C/Y}) = 0$, where $N_{C/Y}$ is the normal bundle. Finally, we refer to [Beh09] for the definition and properties of the Behrend function and of the weighted Euler characteristic.

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2. THE LOCAL MODEL

The global geometry of a fixed smooth curve in a threefold $C \subset Y$ will be analyzed through the local model

$$\mathbb{A}^1 \subset \mathbb{A}^3$$

of a line in affine space. We get started by introducing the moduli space of ideal sheaves for this local model.

Let X be the resolved conifold, i.e. the total space of the rank two bundle $\mathcal{O}_{\mathbb{P}^1}(-1, -1) \rightarrow \mathbb{P}^1$. It is a quasi-projective Calabi-Yau threefold. We let $C_0 \subset X$ be the zero section, and $\mathbb{A}^3 \subset X$ a *fixed* chart of the bundle.

Definition 2.1. For any integer $n \geq 0$, we define

$$M_n \subset I_{n+1}(X, [C_0])$$

to be the open subscheme parametrizing ideal sheaves $\mathcal{I}_Z \subset \mathcal{O}_X$ such that no associated point of Z is contained in $X \setminus \mathbb{A}^3$.

Since C_0 is rigid, we can interpret M_n as the moduli space of “curves” in \mathbb{A}^3 , consisting of a *fixed* affine line $L = C_0 \cap \mathbb{A}^3$ together with n roaming points.

The scheme M_n seems to be the perfect local playground for studying the enumerative geometry of a fixed curve (with n points) in a threefold. Exactly like $\text{Hilb}^n \mathbb{A}^3$ was essential [BF08] to unveil the Donaldson-Thomas theory of $\text{Hilb}^n Y$, where Y is any Calabi-Yau threefold, the space M_n will help us to figure out the DT contribution of a fixed smooth rigid curve in a Calabi-Yau threefold (and, conjecturally, all smooth curves). Forgetting about the Calabi-Yau assumption, we will find out that understanding the local picture in \mathbb{A}^3 gives information about *arbitrary* threefolds, in perfect analogy with the results of [BF08].

In the rest of this section, we show that M_n is isomorphic to the Quot scheme of the ideal sheaf of a line, and we compute its DT invariant via equivariant localization.

Let L denote the line $C_0 \cap \mathbb{A}^3$. Note that if $Z \subset X$ corresponds to a point of M_n , by definition its embedded points can only be supported on L . Similarly, isolated points are confined to the chart $\mathbb{A}^3 \subset X$.

PROPOSITION 2.1. *There is an isomorphism of schemes $M_n \cong \text{Quot}_n(\mathcal{I}_L)$.*

PROOF. Let T be a scheme and let $\iota : \mathbb{A}^3 \times T \rightarrow X \times T$ be the natural open immersion. If $\mathcal{O}_{X \times T} \twoheadrightarrow \mathcal{O}_Z$ represents a T -valued point of M_n , we can consider the sheaf $\mathcal{F} = \mathcal{I}_{C_0 \times T} / \mathcal{I}_Z$, which by definition of M_n is supported on a subscheme of $\mathbb{A}^3 \times T$ which is finite of relative length n over T . Restricting the short exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{C_0 \times T} \rightarrow 0$$

to $\mathbb{A}^3 \times T$ gives a short exact sequence

$$0 \rightarrow \iota^* \mathcal{F} \rightarrow \iota^* \mathcal{O}_Z \rightarrow \mathcal{O}_{L \times T} \rightarrow 0$$

with T -flat kernel, so we get a T -valued point $\mathcal{I}_{L \times T} \twoheadrightarrow \iota^* \mathcal{F}$ of $\text{Quot}_n(\mathcal{I}_L)$, since as we already noticed $\iota^* \mathcal{F}$ has the same support as \mathcal{F} .

Conversely, a T -flat quotient \mathcal{F} of the ideal sheaf $\mathcal{I}_{L \times T}$ determines a flat family of subschemes

$$\mathcal{Z} \subset \mathbb{A}^3 \times T \rightarrow T,$$

where $L \times T \subset \mathcal{Z}$. Taking closures inside $X \times T$, we get closed immersions

$$C_0 \times T \subset \overline{\mathcal{Z}} \subset X \times T.$$

The support of \mathcal{F} is proper over T , and since \mathbb{A}^3 and X are separated, we see that the inclusion maps of $\text{Supp } \mathcal{F}$ in $\mathbb{A}^3 \times T$ and $X \times T$ are proper. This says that the pushforward $\iota_* \mathcal{F}$ is a coherent sheaf on $X \times T$. It agrees with the relative ideal

of the immersion $C_0 \times T \subset \overline{\mathcal{Z}}$, and is supported exactly where \mathcal{F} is. Finally, the short exact sequence

$$0 \rightarrow \iota_* \mathcal{F} \rightarrow \mathcal{O}_{\overline{\mathcal{Z}}} \rightarrow \mathcal{O}_{C_0 \times T} \rightarrow 0$$

says $\mathcal{O}_{\overline{\mathcal{Z}}}$ is T -flat (being an extension of T -flat sheaves), therefore we get a T -valued point of M_n . The two constructions are inverse to each other, whence the claim. \square

Keeping the above result in mind, we will sometimes silently identify M_n with $\text{Quot}_n(\mathcal{I}_L)$, and we will switch from subschemes (or ideal sheaves) to quotient sheaves with no further mention.

Remark 2.1. The resolved conifold X plays little role here. In fact, the above proof shows the following. If there is an immersion $\mathbb{A}^3 \rightarrow Y$ into some Calabi-Yau threefold Y , such that the closure of a line $L \subset \mathbb{A}^3$ becomes a rigid rational curve $C \subset Y$, then the Hilbert scheme $I_{n+1}(Y, [C])$ contains an open subscheme isomorphic to $\text{Quot}_n(\mathcal{I}_L)$.

2.1. The DT invariant. The open subscheme $M_n \subset I_{n+1}(X, [C_0])$ inherits, by restriction, a torus-equivariant symmetric obstruction theory, and therefore a virtual fundamental class

$$[M_n]^{\text{vir}} \in A_0(M_n).$$

The torus \mathbf{T} we are referring to is the two-dimensional torus fixing the Calabi-Yau form on X , and acting on X by rescaling coordinates. We refer the reader to [BB07, § 2.3] for more details on this action and for an accurate description of the fixed locus

$$I_m(X, d[C_0])^{\mathbf{T}} \subset I_m(X, d[C_0])$$

for every $d > 0$. An ideal sheaf $\mathcal{I}_{\mathcal{Z}} \in M_n$ is \mathbf{T} -fixed if it becomes a monomial ideal when restricted to the chosen chart $\mathbb{A}^3 \subset X$. The fixed locus $M_n^{\mathbf{T}} \subset M_n$ is isolated. In the language of the topological vertex, a \mathbf{T} -fixed ideal can be described as a way of stacking n boxes in the corner of the one-legged configuration $(\emptyset, \emptyset, \square)$. We give an example in Figure 1.

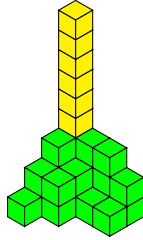


FIGURE 1. A \mathbf{T} -fixed ideal in M_n . The “z-axis” has to be figured as infinitely long, corresponding to the line $L = C_0 \cap \mathbb{A}^3$.

The parity of the tangent spaces at torus-fixed points of $I_m(X, d[C_0])$ was computed in [BB07, Prop. 2.7] (see also the references given there). The result is $(-1)^{m-d}$. Using that $M_n \subset I_{n+1}(X, [C_0])$ is open, we deduce that

$$(-1)^{\dim T_{M_n}|_P} = (-1)^n$$

for all fixed points $P \in M_n^{\mathbf{T}}$. By the symmetry of the obstruction theory, the virtual localization formula [GP99] reads

$$(2.1) \quad [M_n]^{\text{vir}} = (-1)^n [M_n^{\mathbf{T}}].$$

We define the Donaldson-Thomas invariant of M_n by equivariant localization through formula (2.1). Hence we can compute it as

$$\mathrm{DT}(M_n) = (-1)^n \chi(M_n),$$

where $\chi(M_n)$ counts the number of fixed points.

It is easy to see (see for instance the proof of [BB07, Lemma 2.9]) that

$$(2.2) \quad \sum_{n \geq 0} \chi(M_n) q^n = \frac{M(q)}{1 - q}$$

where $M(q) = \prod_{m \geq 1} (1 - q^m)^{-m}$ is the MacMahon function, the generating series of plane partitions. In particular, the DT partition function for the moduli spaces M_n takes the form

$$\sum_{n \geq 0} \mathrm{DT}(M_n) q^{n+1} = q \frac{M(-q)}{1 + q} = q(1 - 2q + 5q^2 - 11q^3 + \dots).$$

In the sum, we have switched indices by one to follow the general convention of weighting the variable q by the holomorphic Euler characteristic.

3. CURVES AND QUOT SCHEMES

3.1. Main characters. Let C be a Cohen-Macaulay curve embedded in a quasi-projective variety Y and let $\mathcal{I}_C \subset \mathcal{O}_Y$ denote its ideal sheaf. For an integer $n \geq 0$, let $Q = \mathrm{Quot}_n(\mathcal{I}_C)$ be the Quot scheme parametrizing 0-dimensional quotients of \mathcal{I}_C , of length n . See [Nit05] for a proof of the representability of the Quot functor in the quasi-projective case. By looking at the full exact sequence

$$0 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{I}_C \rightarrow F \rightarrow 0$$

for a given point $[\mathcal{I}_C \twoheadrightarrow F]$ of Q , we think of the Quot scheme as parametrizing curves $Z \subset Y$ obtained from C , roughly speaking, by adding a finite subscheme of length n .

Definition 3.1. We denote by $W_C^n \subset Q$ the closed subset parametrizing quotients $\mathcal{I}_C \twoheadrightarrow F$ such that $\mathrm{Supp} F \subset C$. We endow it with the reduced scheme structure.

Given a point $[F] \in W_C^n$, the support of F is a subscheme of Y but not of C in general; however, $\mathrm{Supp} F$ is naturally an effective zero-cycle on C . Sending $[F]$ to this cycle is a morphism, as we now show.

LEMMA 3.1. *There is a natural morphism $u : W_C^n \rightarrow \mathrm{Sym}^n C$ sending a quotient to the corresponding zero-cycle.*

PROOF. Let T be a reduced scheme, which we take as the base of a valued point $\mathcal{I}_{C \times T} \twoheadrightarrow \mathcal{F}$ of W_C^n . Let $\pi : Y \times T \rightarrow T$ be the projection. Working locally on Y and T we see that by Nakayama's lemma, $\mathrm{Supp} \mathcal{F} \cap \pi^{-1}(t) = \mathrm{Supp} \mathcal{F}_t$ for every closed point $t \in T$. Then the closed subscheme $\mathrm{Supp} \mathcal{F} \subset Y \times T$ is flat over T (because the Hilbert polynomial of the fibres $\mathrm{Supp} \mathcal{F}_t$ is the constant n and T is reduced), and hence defines a valued point $T \rightarrow \mathrm{Hilb}^n Y$. Composing with the Hilbert-Chow map $\mathrm{Hilb}^n Y \rightarrow \mathrm{Sym}^n Y$ we get a morphism $T \rightarrow \mathrm{Sym}^n Y$ which factors through $\mathrm{Sym}^n C$, by definition of W_C^n . \square

For every partition α of n there is a locally closed subscheme

$$\mathrm{Sym}_{\alpha}^n C \subset \mathrm{Sym}^n C$$

parametrizing zero-cycles with multiplicities dictated by α . These subschemes form a stratification of $\mathrm{Sym}^n C$, which we can use together with the morphism u to stratify W_C^n by locally closed subschemes

$$(3.1) \quad W_C^\alpha = u^{-1}(\mathrm{Sym}_\alpha^n C) \subset W_C^n.$$

In particular, since $\mathrm{Sym}_{(n)}^n C \cong C$, there is a natural morphism

$$(3.2) \quad \pi_C : W_C^{(n)} \rightarrow C$$

corresponding to the deepest stratum.

The main result of this section asserts that, when C is a smooth curve and Y is a smooth threefold, the map (3.2) is a Zariski locally trivial fibration. The proof is based on the Quot scheme adaptation of the results proven by Behrend and Fantechi for $\mathrm{Hilb}^n Y$ [BF08, § 4].

Let us now introduce what will turn out to be the typical fibre of π_C . Recall that X denotes the resolved conifold and $C_0 \subset X$ is the zero section.

Definition 3.2. We denote by $F_n \subset M_n$ the closed subset parametrizing subschemes $Z \subset X$ such that the relative ideal $\mathcal{I}_{C_0}/\mathcal{I}_Z$ is entirely supported at the origin $0 \in L = C_0 \cap \mathbb{A}^3$. We use the shorthand

$$v_n = v_{M_n}|_{F_n}$$

for the restriction of the Behrend function on M_n to F_n .

We can think of F_n and all strata $W_C^\alpha \subset W_C^n$ as endowed with the reduced scheme structure.

Remark 3.1. The morphism $u : W_C^n \rightarrow \mathrm{Sym}^n C$ plays the role of the Hilbert-Chow map $\mathrm{Hilb}^n Y \rightarrow \mathrm{Sym}^n Y$ in the 0-dimensional setting, and the subscheme $F_n \subset M_n$ is the analogue of the punctual Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^3)_0 \subset \mathrm{Hilb}^n \mathbb{A}^3$ parametrizing finite subschemes supported at the origin.

PROPOSITION 3.1. *There is a natural isomorphism $W_L^{(n)} = L \times F_n$. Moreover, if $p : W_L^{(n)} \rightarrow F_n$ is the projection, we have the relation*

$$(3.3) \quad v_{M_n}|_{W_L^{(n)}} = p^* v_n.$$

PROOF. We view L as the additive group G_a and we let it act on itself by translation. This induces an action of L on M_n . Restricting this action to F_n gives a map

$$L \times F_n \rightarrow W_L^{(n)}.$$

This is an isomorphism, whose inverse is the morphism $\pi_L \times \rho : W_L^{(n)} \rightarrow L \times F_n$, where

$$\rho : W_L^{(n)} \rightarrow F_n, \quad Z \mapsto -\pi_L(Z) \cdot Z.$$

The identity (3.3) follows because the Behrend function is constant on orbits and for each $P \in F_n$ the slice $L \times \{P\}$ is isomorphic to an orbit. \square

3.2. Comparing Quot schemes. Let $\varphi : Y \rightarrow Y'$ be a morphism of varieties, where Y is quasi-projective and Y' is complete. Let $C' \subset Y'$ be an integral curve and $C = \varphi^{-1}(C')$ its preimage. Assume both C and $\varphi(C)$ have dimension one. For a fixed integer $n \geq 0$, we let $Q = \mathrm{Quot}_n(\mathcal{I}_C)$ and $Q' = \mathrm{Quot}_n(\mathcal{I}_{C'})$.

We will show how to associate to these data a rational map

$$\Phi : Q \dashrightarrow Q'.$$

The rough idea is that we would like to “push down” the n points in the support of a sheaf $[F] \in Q$ and still get n points, which would ideally form the support of the image sheaf $\varphi_* F$. This only works, as expected, over the open subscheme $V \subset Q$ parametrizing sheaves F such that no two points in $\text{Supp } F \subset Y$ are mapped to the same point in Y' by φ . Moreover, the resulting map $\Phi : V \rightarrow Q'$ turns out to be étale whenever φ is. After extending this result to quasi-projective Y' , we will be able to compare $\text{Quot}_n(\mathcal{I}_C)$ with the local picture of M_n , and pull back (étale-locally) the known results about π_L (Proposition 3.1) to deduce that the maps π_C defined in (3.2) are Zariski locally trivial, at least when C and Y are smooth.

NOTATION. For a scheme S , we will denote $\varphi_S = \varphi \times \text{id}_S : Y \times S \rightarrow Y' \times S$. The case $S = Q$ being quite special, we will let $\tilde{\varphi}$ denote $\varphi_Q = \varphi \times \text{id}_Q$.

Remark 3.2. The curve C' is the scheme-theoretic image of C . Indeed, since C' is reduced, the scheme-theoretic image of $C \rightarrow C'$ is the reduced scheme structure on $\overline{\varphi(C)}$. Since $\varphi(C)$ is not zero-dimensional and C' is irreducible, it follows that $C' = \overline{\varphi(C)}$. In particular, $C' \times S$ is the scheme-theoretic image of $C \times S \subset Y \times S$ under φ_S , for any scheme S .

Remark 3.3. Let \mathcal{E} be the universal sheaf on Q , with support $\Sigma \subset Y \times Q$. Since $\Sigma \rightarrow Q$ is proper (by the very definition of the Quot functor), and it factors through the (separated) projection $\pi : Y' \times Q \rightarrow Q$, necessarily the map $\Sigma \rightarrow Y' \times Q$ must be proper. Since $\tilde{\varphi}_* \mathcal{E}$ is obtained as a pushforward from Σ , it is coherent. Therefore, pushing forward coherent sheaves through the maps φ_S will produce coherent sheaves, even if φ is not proper.

Remark 3.4. Let $[F] \in Q$ be any point, and let $\mathcal{I}_Z \subset \mathcal{I}_C$ be the kernel of the surjection. Then we have closed immersions $C \subset Z \subset Y$ and $C' \subset Z' \subset Y'$, where Z' denotes the scheme-theoretic image of Z . Using that $R^1 \varphi_* F = 0$, we find a commutative diagram of coherent $\mathcal{O}_{Y'}$ -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}_{C'} / \mathcal{I}_{Z'} & \longrightarrow & \mathcal{O}_{Z'} & \longrightarrow & \mathcal{O}_{C'} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \varphi_* F & \longrightarrow & \varphi_* \mathcal{O}_Z & \longrightarrow & \varphi_* \mathcal{O}_C \longrightarrow 0 \end{array}$$

having exact rows. The middle and right vertical arrows are monomorphisms by definition of scheme-theoretic image. For instance,

$$\mathcal{I}_{C'} = \ker(\mathcal{O}_{Y'} \twoheadrightarrow \mathcal{O}_{C'}) = \ker(\mathcal{O}_{Y'} \twoheadrightarrow \mathcal{O}_{C'} \rightarrow \varphi_* \mathcal{O}_C)$$

implies that $\mathcal{O}_{C'} \rightarrow \varphi_* \mathcal{O}_C$ is injective.

The previous remark can be made universal. Let $\mathcal{I}_{C \times Q} \twoheadrightarrow \mathcal{E}$ be the universal quotient, living over $Y \times Q$. Looking at its kernel \mathcal{I}_Z , we get a commutative

diagram

$$\begin{array}{ccccc} C \times Q & \hookrightarrow & Z & \hookrightarrow & Y \times Q \\ \downarrow & & \downarrow & & \downarrow \tilde{\varphi} \\ C' \times Q & \hookrightarrow & Z' & \hookrightarrow & Y' \times Q \end{array}$$

where the horizontal arrows are closed immersions, $\tilde{\varphi} = \varphi \times \text{id}_Q$ and Z' denotes the scheme-theoretic image of Z . We also get a commutative diagram of coherent $\mathcal{O}_{Y' \times Q}$ -modules

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}_{C' \times Q} / \mathcal{I}_{Z'} & \longrightarrow & \mathcal{O}_{Z'} & \longrightarrow & \mathcal{O}_{C' \times Q} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \tilde{\varphi}_* \mathcal{E} & \longrightarrow & \tilde{\varphi}_* \mathcal{O}_Z & \longrightarrow & \tilde{\varphi}_* \mathcal{O}_{C \times Q} \longrightarrow 0 \end{array}$$

having exact rows.

Let us consider the composition

$$(3.4) \quad \alpha : \mathcal{I}_{C' \times Q} \twoheadrightarrow \mathcal{I}_{C' \times Q} / \mathcal{I}_{Z'} \hookrightarrow \tilde{\varphi}_* \mathcal{E}$$

and let us write \mathcal{K} for its cokernel. By Remark 3.3, $\tilde{\varphi}_* \mathcal{E}$ is coherent, hence $\mathcal{K} = \text{coker } \alpha$ is coherent, too. Thus $\text{Supp } \mathcal{K}$ is closed in $Y' \times Q$. Since Y' is complete, the projection $\pi : Y' \times Q \rightarrow Q$ is closed. Therefore the complement

$$(3.5) \quad Q \setminus \pi(\text{Supp } \mathcal{K}) \subset Q$$

is an open subset of Q .

PROPOSITION 3.2. *Let $[F] \in Q$ be a point such that φ is étale in a neighborhood of $\text{Supp } F$ and $\varphi(x) \neq \varphi(y)$ for all distinct points $x, y \in \text{Supp } F$. Then there is an open neighborhood $U \subset Q$ of $[F]$ admitting an étale map $\Phi : U \rightarrow Q'$.*

PROOF. We first observe that we may reduce to prove the result after restricting Y to *any* open neighborhood of $\text{Supp } F$ inside Y . Indeed, if V is any such neighborhood, $\text{Quot}_n(\mathcal{I}_C|_V)$ is an open subscheme of Q that still contains $[F]$ as a point. We will take advantage of this freedom by choosing a suitable V . We divide the proof in two steps.

Step 1: Existence of the map. Let $Z \subset Y$ be the closed subscheme determined by the kernel of $\mathcal{I}_C \twoheadrightarrow F$. Let $Z' \subset Y'$ be its scheme-theoretic image. By our assumption, the length of the quotient $\mathcal{I}_{C'} / \mathcal{I}_{Z'}$ is n . Therefore, the natural monomorphism $\mathcal{I}_{C'} / \mathcal{I}_{Z'} \rightarrow \varphi_* F$ is an isomorphism, so that we get a well-defined point

$$(3.6) \quad [\varphi_* F] \in Q'.$$

Now let $B \subset Y$ denote the support of F and let V be an open neighborhood of B such that φ is étale when restricted to V . We may assume V is affine, and in fact we may also assume $Y = V$, by our initial remark.

In this situation (where we are *not* using that φ is étale yet), we have the Cartesian square

$$\begin{array}{ccc} Y \times [F] & \xrightarrow{i} & Y \times Q \\ \varphi \downarrow & \square & \downarrow \tilde{\varphi} \\ Y' \times [F] & \xrightarrow{j} & Y' \times Q \end{array}$$

where the map $\tilde{\varphi}$ is affine. Therefore, working affine-locally on $Y' \times Q$, we see that the natural base change map $j^* \tilde{\varphi}_* \mathcal{E} \xrightarrow{\sim} \varphi_* F$ is an isomorphism. This proves that the surjection $\mathcal{I}_{C'} \twoheadrightarrow \varphi_* F$ defining the point (3.6) is obtained precisely restricting $\alpha : \mathcal{I}_{C' \times Q} \rightarrow \tilde{\varphi}_* \mathcal{E}$, defined in (3.4), to the slice

$$j : Y' \times [F] \subset Y' \times Q.$$

Letting $U \subset Q$ denote the open subset defined in (3.5), we see that α restricts to a surjection

$$\alpha|_{Y' \times U} : \mathcal{I}_{C' \times U} \twoheadrightarrow \varphi_{U*} \mathcal{E}_U,$$

where $\mathcal{E}_U = \mathcal{E}|_{Y \times U}$. The target is a coherent sheaf, flat over U , and the map restricts to length n quotients

$$\mathcal{I}_{C'} \twoheadrightarrow \varphi_* E,$$

for any closed point $[E] \in U$. Therefore we have just constructed a morphism

$$\Phi : U \rightarrow Q', \quad [E] \mapsto [\varphi_* E].$$

Step 2: Proving it is étale. Now we start using that φ is étale. We may shrink Y further and replace it by any affine open neighborhood of $B = \text{Supp } F$ contained in $Y \setminus A$, where A is the closed subset

$$A = \coprod_{b \in B} \varphi^{-1} \varphi(b) \setminus \{b\} \subset Y.$$

After this choice, the preimage $Y_{\varphi(b)}$ is the single point $\{b\}$, for every $b \in B$. This condition implies that the natural morphism

$$(3.7) \quad \varphi^* \varphi_* F \xrightarrow{\sim} F$$

is an isomorphism. Although this condition is not preserved in any open neighborhood of $[F]$, it is preserved infinitesimally, which is exactly what we need to establish étaleness.

We now use the infinitesimal criterion to show Φ is étale at the point $[F]$. Let $\iota : T \rightarrow \overline{T}$ be a small extension of fat points. Assume we have a commutative square

$$\begin{array}{ccc} T & \xrightarrow{\iota} & \overline{T} \\ g \downarrow & \searrow v & \downarrow h \\ U & \xrightarrow{\Phi} & Q' \end{array}$$

where g sends the closed point $0 \in T$ to $[F]$. Then we want to find a *unique* arrow v making the two induced triangles commutative. Rephrasing this in terms of families of sheaves, let $\mathcal{I}_{C \times T} \twoheadrightarrow \mathcal{G}$ and $\mathcal{I}_{C' \times \overline{T}} \twoheadrightarrow \mathcal{H}$ be the families corresponding to g and h , living over $Y \times T$ and $Y' \times \overline{T}$ respectively. We are after a unique U -valued family $\mathcal{I}_{C \times \overline{T}} \twoheadrightarrow \mathcal{V}$ over $Y \times \overline{T}$ with the following properties.

(\star) The condition $\Phi \circ v = h$ means we can find a commutative diagram

$$\begin{array}{ccc} \mathcal{I}_{C' \times \bar{T}} & \longrightarrow & \varphi_{\bar{T}*} \mathcal{V} \\ \text{id} \downarrow & & \downarrow \simeq \\ \mathcal{I}_{C' \times \bar{T}} & \longrightarrow & \mathcal{H} \end{array} \quad \text{of sheaves on } Y' \times \bar{T}.$$

Let us explain the condition in detail. We use, in the following, the notation $\tilde{p} = 1_Y \times p$ and $\bar{p} = 1_{Y'} \times p$, for a given map p . Looking at the diagram

$$\begin{array}{ccc} Y \times \bar{T} & \xrightarrow{\varphi_{\bar{T}}} & Y' \times \bar{T} \\ \downarrow \tilde{v} & & \downarrow \bar{v} \\ Y \times U & \xrightarrow{\varphi_U} & Y' \times U \\ \downarrow & & \downarrow \Phi \\ Y \times Q & & Y' \times Q' \end{array}$$

we should require

$$\mathcal{H} \cong \bar{v}^* \bar{\Phi}^* \mathcal{E}',$$

where \mathcal{E}' is the universal quotient sheaf on $Y' \times Q'$. However,

$$\bar{v}^* \bar{\Phi}^* \mathcal{E}' \cong \bar{v}^* \varphi_{U*} \mathcal{E}_U \cong \varphi_{\bar{T}*} \mathcal{V},$$

where we have used "affine base change" again.

($\star\star$) Looking at

$$\begin{array}{ccc} Y \times T & \xrightarrow{\varphi_T} & Y' \times T \\ \downarrow \tilde{\iota} & \square & \downarrow \bar{\iota} \\ Y \times \bar{T} & \xrightarrow{\varphi_{\bar{T}}} & Y' \times \bar{T}, \end{array}$$

the condition $v \circ \iota = g$ means we can find a commutative diagram

$$\begin{array}{ccc} \tilde{\iota}^* \mathcal{I}_{C \times \bar{T}} & \longrightarrow & \tilde{\iota}^* \mathcal{V} \\ \text{id} \downarrow & & \downarrow \simeq \\ \mathcal{I}_{C \times T} & \longrightarrow & \mathcal{G} \end{array} \quad \text{of sheaves on } Y \times T.$$

We observe that

- (i) the isomorphism $\varphi_{\bar{T}*} \mathcal{V} \xrightarrow{\sim} \mathcal{H}$ defining (\star), and
- (ii) the isomorphism $\varphi_T^* \varphi_{\bar{T}*} \mathcal{V} \xrightarrow{\sim} \mathcal{V}$, the "infinitesimal thickening" of (3.7),

together determine v uniquely: it is the *unique* arrow corresponding to the isomorphism class of the surjection

$$\mathcal{I}_{C \times \bar{T}} = \varphi_{\bar{T}}^* \mathcal{I}_{C' \times \bar{T}} \twoheadrightarrow \varphi_{\bar{T}}^* \mathcal{H} = \mathcal{V}.$$

To check that condition $(\star\star)$ is fulfilled by this family, we use that $\Phi \circ g = h \circ \iota$. In other words, there is a commutative diagram

$$\begin{array}{ccc} \iota^* \mathcal{I}_{C' \times \bar{T}} & \longrightarrow & \iota^* \mathcal{H} \\ \text{id} \downarrow & & \downarrow \simeq \\ \mathcal{I}_{C' \times T} & \longrightarrow & \varphi_{T*} \mathcal{G} \end{array} \quad \text{of sheaves on } Y' \times T.$$

As before, we have noted that the family corresponding to $\Phi \circ g$ is

$$\bar{g}^* \varphi_{U*} \mathcal{E}_U \cong \varphi_{T*} \mathcal{G},$$

where \bar{g} is the map $\text{id}_{Y'} \times g : Y' \times T \rightarrow Y' \times U$. Now we can compute

$$\iota^* \mathcal{V} = \iota^* \varphi_{\bar{T}}^* \mathcal{H} \cong \varphi_T^* \iota^* \mathcal{H} \cong \varphi_T^* \varphi_{T*} \mathcal{G} \cong \mathcal{G}.$$

This finishes the proof. \square

COROLLARY 3.1. *Let $\varphi : Y \rightarrow Y'$ be an étale map of quasi-projective varieties, $C' \subset Y'$ an integral curve with preimage C . Let $V \subset Q$ be the open subset parametrizing quotients $\mathcal{I}_C \rightarrow F$ such that $\varphi(x) \neq \varphi(y)$ for all $x \neq y \in \text{Supp } F$. Then there is an étale map $\Phi : V \rightarrow Q'$.*

PROOF. To apply Proposition 3.2, we need the target to be complete. Therefore, after completing Y' to a proper variety \bar{Y}' , let us denote by \bar{C}' the scheme-theoretic closure of C' . Then, Proposition 3.2 gives us an étale map $\Phi : V \rightarrow \bar{Q}'$, where the target is the scheme of length n quotients of $\mathcal{I}_{\bar{C}'}$. The map sends $[F] \mapsto [\iota_* \varphi_* F]$, where $\iota : Y' \rightarrow \bar{Y}'$ is the open immersion. However, the support of $\iota_* \varphi_* F$ can be identified with $\text{Supp}(\varphi_* F) \subset Y'$ for all $[F]$, so that Φ actually factors through Q' . \square

3.3. Applications to threefolds. In this section we assume Y and Y' are quasi-projective threefolds. All the other assumptions and notations from the previous sections remain unchanged here.

If $\varphi : Y \rightarrow Y'$ is an étale map, we see that the induced morphism

$$\Phi : V \rightarrow Q'$$

of Corollary 3.1, when restricted to the closed stratum $W_C^{(n)} \subset V$, appears in a Cartesian diagram

$$(3.8) \quad \begin{array}{ccc} W_C^{(n)} & \xrightarrow{\pi_C} & C \\ \Phi \downarrow & \square & \downarrow \varphi \\ W_{C'}^{(n)} & \xrightarrow{\pi_{C'}} & C' \end{array}$$

where the horizontal maps were defined in (3.2). Let $V' \subset Q'$ be the image of the étale map $\Phi : V \rightarrow Q'$. Then the commutative diagram

$$\begin{array}{ccccc} W_C^{(n)} & \hookrightarrow & V & \xrightarrow{\text{open}} & Q \\ \Phi \downarrow & & \downarrow \text{ét} & & \\ W_{C'}^{(n)} & \hookrightarrow & V' & \xrightarrow{\text{open}} & Q' \end{array}$$

yields the relation

$$(3.9) \quad \nu_Q|_{W_C^{(n)}} = \Phi^*(\nu_{Q'}|_{W_{C'}^{(n)}}),$$

which will be useful in the next proof.

PROPOSITION 3.3. *Let $\varphi : Y \rightarrow \mathbb{A}^3$ be an étale map of quasi-projective threefolds, and let $L \subset \mathbb{A}^3$ be a line.*

- (i) *If $C = \varphi^{-1}(L) \subset Y$, we have a natural isomorphism $W_C^{(n)} = C \times F_n$.*
- (ii) *The restricted Behrend function $\nu_Q|_{W_C^{(n)}}$ agrees with the pullback of ν_n under the natural projection to F_n .*

PROOF. With the help of (3.8), we find a diagram

$$\begin{array}{ccccccc} W_C^{(n)} & \xrightarrow{\pi_C} & C & \hookrightarrow & Y & & \\ \Phi \downarrow & & \downarrow & & \downarrow \text{ét} & & \\ F_n & \xleftarrow{p} & W_L^{(n)} & \xrightarrow{\pi_L} & L & \hookrightarrow & \mathbb{A}^3 \end{array}$$

so that the first claim follows by the isomorphism $W_L^{(n)} = L \times F_n$ of Proposition 3.1. As for Behrend functions, we have, using (3.9) and (3.3),

$$\nu_Q|_{W_C^{(n)}} = \Phi^*(\nu_{M_n}|_{W_L^{(n)}}) = \Phi^*(p^*\nu_n).$$

The claim follows. \square

The following can be viewed as the analogue of [BF08, Cor. 4.9].

COROLLARY 3.2. *Let Y be a smooth quasi-projective threefold. If $C \subset Y$ is a smooth curve, the map*

$$\pi_C : W_C^{(n)} \rightarrow C$$

is a Zariski locally trivial fibration with fibre F_n . More precisely, there exists a Zariski open covering $C_i \subset C$ such that for all i one has an isomorphism

$$(3.10) \quad (\pi_C^{-1}(C_i), \nu_Q) \cong (C_i, 1) \times (F_n, \nu_n)$$

of schemes with constructible functions on them.

PROOF. Cover Y with open affine subschemes U_i such that, for each i , the closed immersion $C_i = C \cap U_i \subset U_i$ is given, when C_i is nonempty, by the vanishing of two equations. We can do this because C is a local complete intersection. Possibly after shrinking each U_i , we can find étale maps $U_i \rightarrow \mathbb{A}^3$ and (using the smoothness of C) Cartesian diagrams

$$\begin{array}{ccc} C_i & \hookrightarrow & U_i \\ \downarrow & & \downarrow \text{ét} \\ L & \hookrightarrow & \mathbb{A}^3 \end{array}$$

where L is a fixed line in \mathbb{A}^3 . Combining (3.8) with (both statements of) Proposition 3.3 yields Cartesian diagrams

$$\begin{array}{ccc} C_i \times F_n & \xrightarrow{\pi_{C_i}} & C_i \\ \downarrow & \square & \downarrow \\ W_C^{(n)} & \xrightarrow{\pi_C} & C \end{array}$$

and the claimed decomposition (3.10). \square

4. THE WEIGHTED EULER CHARACTERISTIC OF Q_C^n

The goal of this section is to prove the following result, anticipated in the Introduction.

THEOREM 4.1. *Let Y be a smooth quasi-projective threefold, $C \subset Y$ a smooth curve. If $Q_C^n = \text{Quot}_n(\mathcal{I}_C)$, then*

$$\tilde{\chi}(Q_C^n) = (-1)^n \chi(Q_C^n).$$

4.1. Ingredients in the proof. We briefly discuss the main tools used in the proof of the above formula.

4.1.1. Stratification. We start by observing that we have a stratification

$$(4.1) \quad Q_C^n = \coprod_{\substack{0 \leq j \leq n \\ \alpha \vdash j}} \text{Hilb}^{n-j}(Y \setminus C) \times W_C^\alpha$$

by locally closed subschemes, “separating” the points *away from* the curve from those embedded *on* the curve. We think of a partition $\alpha \vdash j$ as a tuple of positive integers

$$\alpha_1 \geq \cdots \geq \alpha_{r_\alpha} \geq 1$$

such that $\sum \alpha_i = j$. Here r_α is the number of distinct parts of α . Recall that

$$W_C^\alpha \subset Q_C^j,$$

defined for the first time in (3.1), parametrizes configurations of r_α distinct embedded points on C , having respective multiplicities $\alpha_1, \dots, \alpha_{r_\alpha}$. According to (4.1), it is natural to expect the number

$$\tilde{\chi}(Q_C^n) = \chi(Q_C^n, \nu_{Q_C^n})$$

to be computed combining the following data.

First of all, contributions from $\text{Hilb}^{n-j}(Y \setminus C)$ are taken care of by [BF08, Thm. 4.11], which implies the formula

$$(4.2) \quad \tilde{\chi}(\text{Hilb}^k(Y \setminus C)) = (-1)^k \chi(\text{Hilb}^k(Y \setminus C)).$$

Secondly, contributions from $W_C^\alpha \subset W_C^j$ will be fully expressed (thanks to the content of the previous section) in terms of the deepest stratum. The only relevant character here is the “punctual” locus F_n . It will be enough to know that

$$(4.3) \quad \chi(F_j, \nu_j) = (-1)^j \chi(F_j),$$

which follows from [BF08, Cor. 3.5]. Note that here $\chi(F_j) = \chi(M_j)$ counts the number of fixed points of the torus action we have recalled in § 2.1.

4.1.2. *The Behrend function.* According to [Beh09], any complex scheme Z carries a canonical constructible function $\nu_Z : Z \rightarrow \mathbb{Z}$ and one can consider the weighted Euler characteristic

$$\tilde{\chi}(Z) = \chi(Z, \nu_Z) = \sum_{k \in \mathbb{Z}} k \chi(\nu_Z^{-1}(k)).$$

Given a morphism $f : Z \rightarrow X$, one also has the relative weighted Euler characteristic

$$\tilde{\chi}(Z, X) = \chi(Z, f^* \nu_X).$$

We now list its main properties following [Beh09, Prop. 1.8]. First of all, it is clear that $\tilde{\chi}(Z) = \tilde{\chi}(Z, Z)$ through the identity map on Z .

(B1) If $Z = Z_1 \amalg Z_2$ for $Z_i \subset Z$ locally closed, then

$$\tilde{\chi}(Z, X) = \tilde{\chi}(Z_1, X) + \tilde{\chi}(Z_2, X).$$

(B2) Given two morphisms $Z_i \rightarrow X_i, i = 1, 2$, we have

$$\tilde{\chi}(Z_1 \times Z_2, X_1 \times X_2) = \tilde{\chi}(Z_1, X_1) \cdot \tilde{\chi}(Z_2, X_2).$$

(B3) Given a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ W & \longrightarrow & Y \end{array}$$

with $X \rightarrow Y$ smooth and $Z \rightarrow W$ finite étale of degree d , we have

$$\tilde{\chi}(Z, X) = d(-1)^{\dim X/Y} \tilde{\chi}(W, Y).$$

(B4) This is a special case of (B3): if $X \rightarrow Y$ is étale (e.g. an open immersion), then $\tilde{\chi}(Z, X) = \tilde{\chi}(Z, Y)$.

4.2. **The computation.** We can start the proof of Theorem 4.1. Let us shorten $Y_0 = Y \setminus C$ for convenience. After fixing a partition $\alpha \vdash j$, let

$$V_\alpha \subset \prod_i Q_C^{\alpha_i}$$

denote the open subscheme consisting of tuples $(F_1, \dots, F_{r_\alpha})$ of sheaves with pairwise disjoint support. According to Corollary 3.1, we can use the étale cover $\amalg_i Y \rightarrow Y$ to produce an étale morphism

$$f_\alpha : V_\alpha \rightarrow Q_C^j.$$

It is given on points by taking the “union” of the 0-dimensional supports of the sheaves F_i . Letting U_α be the image of f_α , we can form the diagram

$$\begin{array}{ccccc} Z_\alpha & \hookrightarrow & V_\alpha & \xrightarrow{\text{open}} & \prod_i Q_C^{\alpha_i} \\ \text{Galois} \downarrow & & \downarrow f_\alpha & & \\ W_C^\alpha & \hookrightarrow & U_\alpha & \xrightarrow{\text{open}} & Q_C^j \end{array} \quad \square$$

where the Cartesian square defines the scheme Z_α . The morphism on the left is Galois with Galois group G_α , the automorphism group of the partition α . It is easy to see that in fact

$$Z_\alpha = \prod_i W_C^{(\alpha_i)} \setminus \Delta$$

also fits in the Cartesian square

$$(4.4) \quad \begin{array}{ccc} Z_\alpha & \xrightarrow{\text{open}} & \prod_i W_C^{(\alpha_i)} \\ \downarrow & \square & \downarrow \pi_\alpha \\ C^{r_\alpha} \setminus \Delta & \xrightarrow{\text{open}} & C^{r_\alpha} \end{array}$$

where $W_C^{(\alpha_i)} \subset Q_C^{\alpha_i}$ is the deep stratum, Δ denotes the "big diagonal" (where at least two entries are equal), and the vertical map π_α is the product of the fibrations $\pi_C : W_C^{(\alpha_i)} \rightarrow C$, for $i = 1, \dots, r_\alpha$.

We need two identities before we can finish the computation.

First identity. We have

$$(4.5) \quad \chi(W_C^\alpha) = |G_\alpha|^{-1} \chi(C^{r_\alpha} \setminus \Delta) \prod_i \chi(F_{\alpha_i}).$$

Indeed, for each α , the map

$$\pi_\alpha : Z_\alpha \rightarrow C^{r_\alpha} \setminus \Delta$$

appearing in (4.4) is Zariski locally trivial with fiber $\prod_i F_{\alpha_i}$ by Corollary 3.2. Formula (4.5) follows since W_C^α is the free quotient Z_α / G_α .

Second identity. We have

$$(4.6) \quad \tilde{\chi}(Z_\alpha, \prod_i Q_C^{\alpha_i}) = \chi(C^{r_\alpha} \setminus \Delta) \prod_i \chi(F_{\alpha_i}, \nu_{\alpha_i}).$$

Indeed, by Corollary 3.2, we can find a Zariski open cover $\{B_s\}_s$ of $C^{r_\alpha} \setminus \Delta$ such that

$$(\pi_\alpha^{-1} B_s, \nu) \cong (B_s, 1_{B_s}) \times \left(\prod_i F_{\alpha_i}, \prod_i \nu_{\alpha_i} \right).$$

In the left hand side, ν denotes the Behrend function restricted from $\prod_i Q_C^{\alpha_i}$. We can refine this to a locally closed stratification $\coprod_\ell U_\ell = C^{r_\alpha} \setminus \Delta$ such that each U_ℓ is contained in some B_s . Therefore,

$$\begin{aligned} \tilde{\chi}(Z_\alpha, \prod_i Q_C^{\alpha_i}) &= \sum_\ell \tilde{\chi}(\pi_\alpha^{-1} U_\ell, \prod_i Q_C^{\alpha_i}) && \text{by (B1)} \\ &= \sum_\ell \chi(U_\ell \times \prod_i F_{\alpha_i}, 1_{U_\ell} \times \prod_i \nu_{\alpha_i}) \\ &= \sum_\ell \chi(U_\ell, 1_{U_\ell}) \prod_i \chi(F_{\alpha_i}, \nu_{\alpha_i}) && \text{by (B2)} \\ &= \chi(C^{r_\alpha} \setminus \Delta) \prod_i \chi(F_{\alpha_i}, \nu_{\alpha_i}), \end{aligned}$$

and (4.6) is proved.

Note that combining (4.1) and (4.5) we get

$$(4.7) \quad \chi(Q_C^n) = \sum_{j, \alpha} \chi(\text{Hilb}^{n-j} Y_0) \cdot |G_\alpha|^{-1} \chi(C^{r_\alpha} \setminus \Delta) \prod_i \chi(F_{\alpha_i}).$$

We now have all the tools to finish the computation. Let us fix j and a partition $\alpha \vdash j$. We define

$$D_\alpha \subset \mathrm{Hilb}^{n-j} Y \times \prod_i Q_C^{\alpha_i}$$

to be the set of tuples $(Z_0, F_1, \dots, F_{r_\alpha})$ such that $(F_1, \dots, F_{r_\alpha}) \in V_\alpha$ and the support of Z_0 does not meet the support of any F_i . Then D_α is an open subscheme. The Galois cover $1 \times f_\alpha : \mathrm{Hilb}^{n-j} Y_0 \times Z_\alpha \rightarrow \mathrm{Hilb}^{n-j} Y_0 \times W_C^\alpha$ extends to an étale map $D_\alpha \rightarrow Q_C^n$, so that we have a commutative diagram

$$(4.8) \quad \begin{array}{ccc} \mathrm{Hilb}^{n-j} Y_0 \times Z_\alpha & \hookrightarrow & D_\alpha \\ \downarrow 1 \times f_\alpha & & \downarrow \text{ét} \\ \mathrm{Hilb}^{n-j} Y_0 \times W_C^\alpha & \hookrightarrow & Q_C^n. \end{array}$$

Therefore we can start computing $\tilde{\chi}(Q_C^n) = \chi(Q_C^n, \nu_{Q_C^n})$ as follows:

$$\begin{aligned} \tilde{\chi}(Q_C^n) &= \sum_{j, \alpha} \tilde{\chi}(\mathrm{Hilb}^{n-j} Y_0 \times W_C^\alpha, Q_C^n) && \text{by (B1) applied to (4.1)} \\ &= \sum_{j, \alpha} |G_\alpha|^{-1} \tilde{\chi}(\mathrm{Hilb}^{n-j} Y_0 \times Z_\alpha, D_\alpha) && \text{by (B3) applied to (4.8)} \\ &= \sum_{j, \alpha} |G_\alpha|^{-1} \tilde{\chi}(\mathrm{Hilb}^{n-j} Y_0 \times Z_\alpha, \mathrm{Hilb}^{n-j} Y \times \prod_i Q_C^{\alpha_i}) && \text{by (B4)} \\ &= \sum_{j, \alpha} |G_\alpha|^{-1} \tilde{\chi}(\mathrm{Hilb}^{n-j} Y_0, \mathrm{Hilb}^{n-j} Y) \cdot \tilde{\chi}(Z_\alpha, \prod_i Q_C^{\alpha_i}) && \text{by (B2)} \\ &= \sum_{j, \alpha} |G_\alpha|^{-1} \tilde{\chi}(\mathrm{Hilb}^{n-j} Y_0) \cdot \chi(C^{r_\alpha} \setminus \Delta) \prod_i \chi(F_{\alpha_i}, \nu_{\alpha_i}) && \text{by (B4) and (4.6)} \\ &= (-1)^n \sum_{j, \alpha} \chi(\mathrm{Hilb}^{n-j} Y_0) \cdot |G_\alpha|^{-1} \chi(C^{r_\alpha} \setminus \Delta) \prod_i \chi(F_{\alpha_i}) && \text{by (4.2) and (4.3)} \\ &= (-1)^n \chi(Q_C^n) && \text{by (4.7).} \end{aligned}$$

This completes the proof of Theorem 4.1.

QUESTION 4.1. It would be nice to determine whether the Behrend function on $M_n = \mathrm{Quot}_n(\mathcal{S}_L)$ is the constant sign $(-1)^n$. As far as we know, this is still open even when the curve is absent, i.e. for $\mathrm{Hilb}^n \mathbb{A}^3$.

5. IDEALS, PAIRS AND QUOTIENTS

In this section we give some applications of the formula

$$\tilde{\chi}(Q_C^n) = (-1)^n \chi(Q_C^n).$$

We show that the DT/PT correspondence holds for the contribution of a smooth *rigid* curve in a projective Calabi-Yau threefold. We discuss, at a conjectural level, the case of an arbitrary smooth curve.

5.1. Local contributions. We fix a smooth projective threefold Y and a Cohen-Macaulay curve $C \subset Y$ of arithmetic genus $g = 1 - \chi(\mathcal{O}_C)$, embedded in class $\beta \in H_2(Y, \mathbb{Z})$. We will use the Quot scheme to endow the closed subset

$$\{Z \subset Y \mid C \subset Z, \chi(\mathcal{S}_C / \mathcal{S}_Z) = n\} \subset I_{1-g+n}(Y, \beta)$$

with a natural scheme structure.

LEMMA 5.1. *There is a closed immersion $\iota : Q_C^n \rightarrow I_{1-g+n}(Y, \beta)$.*

PROOF. Let $\mathcal{I}_{C \times T} \twoheadrightarrow \mathcal{F}$ be a flat family of quotients parametrized by a scheme T . Letting $Z \subset Y \times T$ be the subscheme defined by the kernel of the surjection, we get an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{C \times T} \rightarrow 0.$$

The middle term is flat over T , therefore it determines a point in the Hilbert scheme of Y . The discrete invariants β and $\chi = 1 - g + n$ are the right ones, as can be seen by restricting the above short exact sequence to closed points of T . Therefore we get a morphism

$$\iota : Q_C^n \rightarrow I_{1-g+n}(Y, \beta).$$

The correspondence at the level of functor of points is injective, and the morphism is proper (since the Quot scheme is proper, as Y is projective). Therefore ι is a closed immersion. \square

Definition 5.1. We define

$$(5.1) \quad I_n(Y, C) \subset I_{1-g+n}(Y, \beta)$$

to be the scheme-theoretic image of $\iota : Q_C^n \rightarrow I_{1-g+n}(Y, \beta)$.

Remark 5.1. The closed subset $|I_n(Y, C)| \subset I_{1-g+n}(Y, \beta)$ also has a scheme structure induced by GIT wall-crossing [ST11]. Another scheme structure is defined in the recent paper [BK16]. See in particular Definition 4, where the notation used is $\text{Hilb}^n(Y, C)$. We believe both these scheme structures agree with the one of our Definition 5.1, in which case they describe schemes isomorphic to Q_C^n .

Assume Y is a projective Calabi-Yau threefold. By the main result of [Beh09], the degree β curve counting invariants

$$\text{DT}_{m, \beta} = \int_{[I_m(Y, \beta)]^{\text{vir}}} 1, \quad \text{PT}_{m, \beta} = \int_{[P_m(Y, \beta)]^{\text{vir}}} 1$$

can be computed as weighted Euler characteristics of the corresponding moduli spaces, since the obstruction theories defining the virtual cycles are symmetric. One can define the contribution of C to the above invariants as

$$(5.2) \quad \text{DT}_{n, C} = \chi(I_n(Y, C), \nu_I), \quad \text{PT}_{n, C} = \chi(P_n(Y, C), \nu_P).$$

Here we have set $I = I_{1-g+n}(Y, \beta)$ and $P = P_{1-g+n}(Y, \beta)$. The subscheme $P_n(Y, C) \subset P$ consists of stable pairs with Cohen-Macaulay support equal to C . Note that these integers remember how C sits inside Y , since the weight is the Behrend function coming from the full moduli space.

An immediate consequence of Theorem 4.1 is a formula for the DT contribution of a smooth rigid curve.

THEOREM 5.1. *Let Y be a projective Calabi-Yau threefold, $C \subset Y$ a smooth rigid curve. Then*

$$\text{DT}_{n, C} = (-1)^n \chi(I_n(Y, C)).$$

PROOF. The inclusion (5.1) is both open and closed thanks to the infinitesimal isolation of C . Then $\nu_I|_{I_n(Y, C)} = \nu_{I_n(Y, C)}$, thus

$$\text{DT}_{n, C} = \tilde{\chi}(I_n(Y, C)) = (-1)^n \chi(I_n(Y, C)),$$

as claimed. \square

Remark 5.2. In the rigid case, $\text{DT}_{n,C}$ is a DT invariant in the classical sense, namely it is the degree of the virtual class $[I_n(Y, C)]^{\text{vir}}$ obtained by restricting the one on $I_{1-g+n}(Y, \beta)$.

Theorem 5.1 can be seen as an instance of the following more general result, which is also a direct consequence of Theorem 4.1.

PROPOSITION 5.1. *Let Y be a smooth projective threefold. If $C \subset Y$ is a smooth curve of genus g , then*

$$(5.3) \quad \sum_{n \geq 0} \tilde{\chi}(I_n(Y, C)) q^n = M(-q)^{\chi(Y)} (1+q)^{2g-2}.$$

PROOF. For any smooth threefold X we have Cheah's formula [Che96]

$$\sum_{n \geq 0} \chi(\text{Hilb}^n X) q^n = M(q)^{\chi(X)}.$$

We use this with $X = Y_0 = Y \setminus C$, together with formula (4.7), to compute

$$\begin{aligned} \sum_{n \geq 0} \chi(I_n(Y, C)) q^n &= M(q)^{\chi(Y \setminus C)} \cdot \left(\sum_{n \geq 0} \chi(F_n) q^n \right)^{\chi(C)} \\ &= M(q)^{\chi(Y \setminus C)} \cdot \left(\sum_{n \geq 0} \chi(M_n) q^n \right)^{\chi(C)} \\ &= M(q)^{\chi(Y \setminus C)} \cdot \left(\frac{M(q)}{1-q} \right)^{\chi(C)} \quad \text{by (2.2)} \\ &= M(q)^{\chi(Y)} (1-q)^{2g-2}. \end{aligned}$$

The claimed formula follows by Theorem 4.1. □

Remark 5.3. Formula (5.3) can be rewritten as

$$(5.4) \quad \sum_{n \geq 0} \tilde{\chi}(I_n(Y, C)) q^n = M(-q)^{\chi(Y)} \sum_{n \geq 0} \tilde{\chi}(P_n(Y, C)) q^n.$$

Indeed $P_n(Y, C) = \text{Sym}^n C$ is smooth of dimension n , thus $\tilde{\chi} = (-1)^n \chi$. The latter identity can be seen as the ν -weighted version of the "local" wall-crossing formula between ideals and stable pairs, which was already established for a single Cohen-Macaulay curve at the level of Euler characteristics [ST11, Thm. 1.5]. In other words, (5.4) is precisely what happens to the Stoppa-Thomas identity

$$\sum_{n \geq 0} \chi(I_n(Y, C)) q^n = M(q)^{\chi(Y)} \sum_{n \geq 0} \chi(P_n(Y, C)) q^n$$

when we replace q by $-q$.

5.2. DT/PT wall-crossing at a single curve. Let C be a smooth curve of genus g , embedded in class β in a smooth projective Calabi-Yau threefold Y . Let us define the generating series

$$\begin{aligned} \text{DT}_C(q) &= \sum_{n \geq 0} \text{DT}_{n,C} q^n \\ \text{PT}_C(q) &= \sum_{n \geq 0} \text{PT}_{n,C} q^n \end{aligned}$$

encoding the local contributions defined in (5.2). The stable pair side has already been computed [PT10, Lemma 3.4]. The result is

$$(5.5) \quad \text{PT}_C(q) = n_{g,C} \cdot (1+q)^{2g-2},$$

where $n_{g,C}$ is the g -th BPS number of C . For instance, if C is rigid, then $n_{g,C} = 1$ and thanks to Theorem 5.1 we see that (5.3) can be rewritten as

$$\mathrm{DT}_C(q) = M(-q)^{\chi(Y)} \cdot \mathrm{PT}_C(q).$$

This formula can be seen as a “local DT/PT correspondence”, or local wall-crossing formula at C . We next prove that such formula, for arbitrary C , is equivalent to the following conjecture.

Conjecture 1. Let C be a smooth curve in a projective Calabi-Yau threefold Y . Let $\mathcal{I} = I_{1-g}(Y, \beta)$ be the Hilbert scheme where the ideal sheaf of C lives as a point. Then, for all n , one has

$$\mathrm{DT}_{n,C} = v_{\mathcal{I}}(\mathcal{I}_C) \cdot \tilde{\chi}(I_n(Y, C)).$$

Remark 5.4. An equivalent formula was conjectured by Bryan and Kool in their recent paper [BK16]. See Conjecture 18 in *loc. cit.* for the precise (more general) setting.

THEOREM 5.2. *Let Y be a projective Calabi-Yau threefold, $C \subset Y$ a smooth curve. Then Conjecture 1 is equivalent to the wall-crossing identity*

$$\mathrm{DT}_C(q) = M(-q)^{\chi(Y)} \cdot \mathrm{PT}_C(q).$$

PROOF. Combining (5.5) with (5.3), we see that the right hand side of the formula equals

$$n_{g,C} \cdot \sum_{n \geq 0} \tilde{\chi}(I_n(Y, C)) q^n.$$

Therefore the DT/PT correspondence holds at C if and only if

$$\mathrm{DT}_{n,C} = n_{g,C} \cdot \tilde{\chi}(I_n(Y, C)).$$

We are then left with proving that $v_{\mathcal{I}}(\mathcal{I}_C) = n_{g,C}$. Recall that the moduli space of ideal sheaves is isomorphic to the moduli space of stable pairs along the *open* subschemes parametrizing *pure* curves. Moreover, the map $\phi : P_{1-g}(Y, \beta) \rightarrow \mathcal{M}$ to the moduli space of stable pure sheaves considered in [PT10], defined by forgetting the section of a stable pair, satisfies the relation

$$v_{P_{1-g}(Y, \beta)} = (-1)^g \phi^* v_{\mathcal{M}}$$

by [PT10, Thm. 4]. Hence

$$\begin{aligned} v_{\mathcal{I}}(\mathcal{I}_C) &= v_{\mathcal{I}^{\mathrm{pur}}}(\mathcal{I}_C) \\ &= v_{P_{1-g}(Y, \beta)}([\mathcal{O}_Y \twoheadrightarrow \mathcal{O}_C]) \\ &= (-1)^g v_{\mathcal{M}}(\mathcal{O}_C) \\ &= n_{g,C} \end{aligned}$$

where the last equality is [PT10, Prop. 3.6]. \square

Remark 5.5. Thanks to the identity $v_{\mathcal{I}}(\mathcal{I}_C) = n_{g,C}$, proved in the course of Theorem 5.2, Conjecture 1 can be rephrased as

$$\mathrm{DT}_{n,C} = v_P|_{P_n(Y, C)} \cdot \chi(I_n(Y, C)),$$

where $v_P|_{P_n(Y, C)}$ is the constant $(-1)^n \cdot n_{g,C} = (-1)^{n-g} v_{\mathcal{M}}(\mathcal{O}_C)$. In particular the conjecture says that the DT and PT contributions of C differ from the Euler characteristic of the corresponding moduli space by the *same* constant.

We end the paper with some speculations, indicating plausibility reasons why Conjecture 1 should hold true.

Suppose we were able to show that, given a point $\mathcal{J}_Z \in I_n(Y, C) \subset I$, a formal neighborhood of \mathcal{J}_Z in I is isomorphic to a product

$$U \times V,$$

where U is a formal neighborhood of \mathcal{J}_C in \mathcal{I} and V is a formal neighborhood of \mathcal{J}_Z in $I_n(Y, C)$. Then, since the Behrend function value $\nu(P)$ only depends on a formal neighborhood of P [Jia], this would immediately lead to the Behrend function identity

$$(5.6) \quad \nu_I|_{I_n(Y, C)} = \nu_{\mathcal{I}}(\mathcal{J}_C) \cdot \nu_{I_n(Y, C)},$$

from which Conjecture 1 follows after integration. One reason to believe in a product decomposition as above is the following. At least when the maximal purely 1-dimensional part $C \subset Z$ is smooth, one may expect to be able to “separate” infinitesimal deformations of C (the factor U) from those deformations of Z that keep C fixed (the factor V in the Quot scheme). This decomposition is manifestly false when C acquires a singularity, and we do not know of any counterexample in the smooth case.

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KJELL ARHOLMS 41, 4021 STAVANGER (NORWAY)

E-mail address: andrea.ricolfi@uis.no